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We study T^3 Gowdy spacetimes in the Einstein-Maxwell-dilaton-axion system and show by the Fuchsian algorithm that they have in general asymptotically velocity-term dominated singularities. The families of the corresponding solutions depend on the maximum number of arbitrary functions. Although coupling of the dilaton field with the Maxwell and/or the axion fields corresponds to the “potential” which appears in the Hamiltonian of vacuum Bianchi IX spacetimes, our result means that the spacetimes do not become Mixmaster necessarily.

I. INTRODUCTION

The singularity theorem gives us some sufficient conditions for the existence of spacetime singularities (a spacetime with incomplete geodesics) [18,17]. These conditions fall into four categories : (1) the strong energy condition; (2) the generic condition; (3) the chronology condition and (4) the existence of trapped regions. In addition, after the discovery of the singularity theorem, various generalization by replacing these conditions have been suggested by some authors (see e.g. Refs. [32,24,30]). So, although, we have new many variations of the singularity theorem, none of them gives information on the nature of the singularity.

To the best of our knowledge, the first research about the nature of the singularity was done by Belinskii, Khalatnikov and Lifshitz (BKL) [6]. They investigated how the spacetime evolves into the singularity and conjectured that the dynamics of nearby observers would decouple near singularities. (the *BKL conjecture*.) BKL described a vacuum and spatially homogeneous (mainly Bianchi IX) spacetime singularity as an infinite sequence of Kasner epochs (“oscillatory approach to the singularity”). Independently, Misner showed the same behavior in terms of exponentially growing “potential” terms constructed by the spatial curvature. The spacetime is bounced by the potentials a infinite number of times. Such behavior is called the *Mixmaster dynamics* [27]. Then, BKL further speculated that a generic singularity should be locally behaved like a Mixmaster type. To date, the BKL conjecture has neither been validated nor invalidated in general by rigorous arguments.

There is a special case of the BKL conjecture called the *asymptotically velocity-term dominated* (AVTD) singularity. It is not described by a infinite sequence of Kasner epochs but by only one epoch (i.e., “no oscillatory approach to the singularity”) [20,34]. In brief, the characteristic feature of the AVTD singular behavior is that all spatial derivative terms of the field equations are negligible sufficiently close to the singularity. Hence, Kasner spacetimes are necessarily (A)VTD. It is possible to make rigorous arguments on the nature of the AVTD singularity since the AVTD solutions are simpler than the Mixmaster solutions in the sense that there is no complicated oscillation. Our interest is to verify BKL conjecture and clarify whether spacetimes is AVTD or not in non-vacuum and/or spatially inhomogeneous cases.

In order to attack the above issue, we will consider Gowdy spacetimes [15]. They are spatially compact spacetimes which have $U(1) \times U(1)$ symmetry and vanishing twist. Gowdy spacetimes are adopted in various investigations because they are one of the most manageable inhomogeneous spacetimes and can give essential features of inhomogeneity in spite of its simpleness. The behavior near the singularity in Gowdy spacetime have also been discussed in various situations. In the vacuum case, Isenberg and Moncrief have shown that polarized Gowdy spacetimes are AVTD [20]. Grubišić and Moncrief [16] and Kichenassamy and Rendall [22] have shown that non-polarized T^3 Gowdy spacetimes have AVTD singularities in generic in the sense that the AVTD singular solutions to Einstein’s equations depend on the maximal number of arbitrary functions if the *low velocity condition* is satisfied. When one of these functions is constant (i.e. non-generic), it was also shown without the low velocity condition that non-polarized T^3 Gowdy spacetimes are AVTD [22]. Using numerical method, Berger, Garfinkle and Moncrief found that non-polarized T^3 Gowdy spacetimes have AVTD singularities even in the situation that the low velocity condition is broken initially [10,9,7]. Recently, Garfinkle has shown that $S^2 \times S^1$ Gowdy spacetimes have AVTD singularities numerically [14].

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AVTD solutions with matter fields present have been found. Such examples include polarized Gowdy spacetimes admitting scalar fields minimally coupled with the Maxwell field [11] and the Einstein-dilaton-axion (EDA) system in polarized Gowdy spacetimes [4]. It has been shown numerically, however, that magnetic Gowdy spacetimes are not AVTD but Mixmaster [7,35]. Hence, it is nontrivial to determine whether the spacetime is Mixmaster or AVTD type. Because of these facts, we shall study the non-vacuum and spatially inhomogeneous case.

There is another point which needs to be clarified. Belinskii and Khalatnikov have suggested that the existence of massless scalar fields suppresses Mixmaster behavior by examining the Bianchi I spacetimes [5]. It is due to the fact that the scalar field is algebraically equivalent to the stiff matter in some cases. Recently, Berger found some confirmation for this suggestion by numerical analysis [8]. In both papers, however, it was suggested that exponential coupling of the scalar field to the Maxwell field restores Mixmaster behavior which was suppressed initially by the scalar field [5,8]. It is important to note that such matter fields arise naturally from low energy effective superstring theory [33], i.e. the Einstein-Maxwell-dilaton-axion (EMDA) system. The EMDA system has been discussed actively in the context of the black hole solutions and singularities, and gives characteristic features different from the Einstein-Maxwell system [25].

Our purpose in this paper is to explore the nature of singularities in the EMDA system with Gowdy spacetimes on $T^3 \times \mathbf{R}$. We make use of the Fuchsian algorithm developed by Kichenassamy and Rendall [22]. Although the system treated here is very complicated, the Fuchsian algorithm is powerful enough for the system since this algorithm is independent of non-linearity, number of functions and dimension of space. We will show that there exist families of AVTD singular solutions of the EMDA field equations, which are generic in the sense that they depend on the maximal number of arbitrary functions. Hence, our result shows that exponential coupling of the scalar field to the Maxwell field does not necessarily restore the Mixmaster behavior. The EMDA system with Gowdy spacetimes is not so simple as Berger suggested.

Organization of this paper is as follows. In Sec. II, an appropriate definition of AVTD is given. In Sec. III, we discuss the importance of studying the EMDA system. Sec. IV is devoted to introducing T^3 Gowdy spacetimes in the EMDA system. Sec. V presents a brief review of the Fuchsian algorithm. Our main results are given in Sec. VI. Finally, Sec. VII is a summary.

II. ASYMPTOTICALLY VELOCITY-TERM DOMINATED (AVTD) SINGULARITY

In this section, we give precise definition of AVTD. In contrast to the Mixmaster singularity which is complicated, the AVTD singularity is called simple or quiescent [13,20]. An essential difference between Mixmaster and AVTD singularities is whether spatial curvature terms in Einstein's equations becomes dominant or not as the system approaches the singularities. As mentioned in the previous section, the “potential” terms constructed by the spatial curvatures grow exponentially in the Mixmaster case. These “potential” terms give rise to an infinite number of reflection and the spacetime will approach an oscillating state, i.e. Mixmaster singularities described as an infinite sequence of Kasner epochs. On the other hand, such a “potential” term does not exist in the AVTD case. Therefore, reflection by the potential does not occur and the spacetime will approach a single Kasner epoch.

Now, let us define AVTD. Suppose that a four-dimensional spacetime (M, g) with the signature $(-+++)$ satisfies Einstein's equations, $G_{\mu\nu} = T_{\mu\nu}$, where $G_{\mu\nu} \equiv {}^{(4)}R_{\mu\nu} - \frac{1}{2} {}^{(4)}R g_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor. For any $3+1$ decomposition, we can obtain the constraint equations,

$${}^{(3)}R - K^{ab}K_{ab} + (trK)^2 = 2T_0^0, \quad (2.1)$$

$${}^{(3)}\nabla_a K_b^a - {}^{(3)}\nabla_b (trK) = -T_b^0, \quad (2.2)$$

and the evolution equations,

$$\partial_t h_{ab} = -2N K_{ab} + \mathcal{L}_{\mathbf{N}} h_{ab}, \quad (2.3)$$

$$\partial_t K_b^a = N \left[{}^{(3)}R_b^a + (trK) K_b^a \right] - {}^{(3)}\nabla_b {}^{(3)}\nabla^a N + \mathcal{L}_{\mathbf{N}} K_b^a + 2N \left[T_b^a + \frac{1}{2} h_b^a (T_0^0 - T_c^c) \right], \quad (2.4)$$

where h_{ab} , K_{ab} and (trK) are the first and second fundamental form and mean curvature of the 3-dimensional spacelike hypersurface, respectively. Also, ${}^{(3)}\nabla$, ${}^{(3)}R_{ab}$ and ${}^{(3)}R$ are the spatial covariant derivative, spatial Ricci and scalar curvature, respectively. N and \mathbf{N} are the lapse function and shift vector. $\mathcal{L}_{\mathbf{N}}$ is the Lie derivative along the vector field \mathbf{N} . From the full Einstein equations, we define the velocity term dominated (VTD) equations as follows:

$$-K^{ab}K_{ab} + (trK)^2 = 2\tilde{T}_0^0, \quad (2.5)$$

$${}^{(3)}\nabla_a K_b^a - {}^{(3)}\nabla_b (trK) = -\tilde{T}_b^0, \quad (2.6)$$

$$\partial_t h_{ab} = -2NK_{ab}, \quad (2.7)$$

$$\partial_t K_b^a = N[(trK)K_b^a] + 2N\left[\tilde{T}_b^a + \frac{1}{2}h_b^a(\tilde{T}_0^0 - \tilde{T}_c^c)\right], \quad (2.8)$$

where $\tilde{\bullet}$ is modification of \bullet . (Generally, the spatial derivatives are removed.)

In this setting we can give a definition of an AVTD spacetime and its singularity.

Definition 1 (Isenberg and Moncrief [20]) *A spacetime (M, g) is AVTD if*

1. *it is a solution to Einstein's equations;*
2. *there exists another (M, \tilde{g}) and there exists a foliation $i_t : \Sigma^3 \rightarrow M$ such that*
 - (a) *(M, \tilde{g}) obeys the VTD equations relative to i_t ;*
 - (b) *metric g asymptotically approaches \tilde{g} as $t \rightarrow 0$ in the sense that, for an appropriate norm $\|\bullet\|$ on the space of $3+1$ quantities \bullet , for any $\epsilon > 0$ there exists $T > 0$ such that $\|\bullet - \tilde{\bullet}\| < \epsilon$ for all $t < T$.*

Furthermore, a spacetime (M, g) has the AVTD singularity if

1. *it contains a spacelike singularity at $t \rightarrow 0$;*
2. *it is AVTD.*

To sum up, a singularity is called AVTD if all spatial curvature and spatial derivative terms in Einstein's equations can be neglected near the singularity $t \rightarrow 0$. Then, near the AVTD singularity, AVTD solutions can be interpreted as a different spatially homogeneous cosmology at each point in space.

In order to see an example of the AVTD singularity, let us consider vacuum Bianchi I (Kasner) spacetimes. Since these spacetimes are spatially homogeneous, there is no spatial derivative terms. Furthermore, they are spatially flat, i.e. ${}^{(3)}R = 0$. Therefore, the singularity is necessarily AVTD. On the other hand, Bianchi IX spacetimes are not AVTD but Mixmaster [6,27] since the spatial curvature does not vanish and the “potential” term constructed by the spatial curvature cannot be neglected as approaching to the singularity (See Sec. III).

III. INFLUENCE OF MATTER FIELDS

BKL assumed the vacuum spacetimes originally in the investigation of the singularity. The reason for putting such assumption is illustrated as follows. Consider class A Bianchi spacetimes with the metric

$$ds^2 = -dt^2 + e^{2\Omega}(e^{2\beta})_{ij}\sigma^i\sigma^j,$$

where e^Ω is the averaged scale factor, $\beta = \text{diag}(-2\beta_+, \beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-)$ is a traceless matrix that determines the anisotropy and $d\sigma^i = \epsilon^i_{jk}\sigma^j \wedge \sigma^k$. For simplicity, take $\sigma^i = dx^i$, i.e. Bianchi I spacetimes. Then, $e^{3\Omega} = t$ and $\Omega^2 = \beta_+^2 + \beta_-^2$. The spacetimes have singularities at $\Omega \rightarrow -\infty$. The square of shear σ is estimated as $\sigma^2 \sim t^{-2} \sim e^{-6\Omega}$. Further, consider a perfect fluid whose equation of state is $p = (\gamma - 1)\rho$, where ρ and p are the energy density and the pressure, respectively. The contribution from the shear is more dominant than that of the energy density $\rho \sim e^{-3\Omega\gamma}$ near singularities ($\Omega \rightarrow -\infty$) if $1 \leq \gamma < 2$ for natural matters such as a dust ($\gamma = 1$) and a radiation ($\gamma = 4/3$). Thus, the effect of the matter field is negligible for the behavior of the spacetime near singularities, and we have only to consider the vacuum case. Indeed, the known Bianchi perfect fluid solutions have similar behavior as the vacuum solutions near the singularities [34].

We must, however, look more carefully into the influence of the energy density in the case $\gamma = 2$, where time dependence of the energy density and shear are contribute equally. The matter with $\gamma = 2$ is a stiff matter [3], and is not realistic matter in the relativistic sense. However, it is known that a scalar field ϕ is algebraically equivalent

to a stiff matter if $\nabla\phi$ is timelike [34], where ∇ is covariant derivative with respect to g . Therefore, the existence of scalar fields will give strong influence on the nature of singularities.

Actually, Belinskii and Khalatnikov found that a massless scalar field can suppress Mixmaster oscillations [5]. The result obtained by Belinskii and Khalatnikov is that all of Kasner exponents can be positive in Bianchi I spacetimes if the scalar field exists. Therefore, complicated permutation in the direction of anisotropic contraction toward singularities does not appear. Numerically, Berger confirmed Belinskii and Khalatnikov's result in Bianchi IX spacetimes and furthermore, she showed that Mixmaster oscillations are suppressed in non-polarized $U(1)$ -symmetric and magnetic Gowdy spacetimes which are Mixmaster if there exist no scalar fields [8].

On the contrary, there is a possibility that the existence of exponential potentials of scalar fields drastically changes the behavior of singularities, i.e. the Mixmaster dynamics may come back. Recall the mechanism of the Mixmaster dynamics. The Hamiltonian \mathcal{H}_{IX} for Bianchi IX spacetimes [27,8] is

$$2\mathcal{H}_{IX} = -p_\Omega^2 + p_+^2 + p_-^2 + V(\Omega, \beta_\pm),$$

where p_Ω and p_\pm are conjugate momenta of Ω and β_\pm , respectively. The “potential” term $V(\Omega, \beta_\pm)$ is given by

$$V(\Omega, \beta_\pm) = e^{4\Omega} \left[e^{-8\beta_+} + e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} - 2 \left(e^{4\beta_+} + e^{-2\beta_+ - 2\sqrt{3}\beta_-} + e^{-2\beta_+ + 2\sqrt{3}\beta_-} \right) \right].$$

In the Bianchi I spacetimes there is no such “potential” term. It is known that Bianchi IX spacetimes have a singularity as $\Omega \rightarrow -\infty$. If the spacetimes are AVTD, the “potential” term must vanish as $\Omega \rightarrow -\infty$. When $V(\Omega, \beta_\pm) = 0$, we have $\beta_\pm = \beta_\pm^0 + v_\pm|\Omega|$, $p_\Omega = \text{const.}$ and $p_\pm = \text{const.}$ by varying the Hamiltonian \mathcal{H}_{IX} , where $v_\pm \equiv p_\pm/|p_\Omega|$ and β_\pm^0 are constant. The Hamiltonian constraint $\mathcal{H}_{IX} = 0$ is $v_+^2 + v_-^2 = 1$. Then,

$$V \approx e^{-4|\Omega|(1+2\cos\theta)} + e^{-4|\Omega|(1-\cos\theta-\sqrt{3}\sin\theta)} + e^{-4|\Omega|(1-\cos\theta+\sqrt{3}\sin\theta)},$$

where we parameterize $v_+ = \cos\theta$ and $v_- = \sin\theta$. For generic θ (except for $\theta = 0, 2\pi/3, 4\pi/3$), the term V exponentially grows as $\Omega \rightarrow -\infty$. This is a contradiction to vanish the “potential” term as $\Omega \rightarrow -\infty$. Thus, Bianchi IX spacetimes are not AVTD since the “potential” term becomes dominant near the singularity. As a result, reflections by the potential cause the Mixmaster dynamics.

As we have seen in the above discussion, the essential point is the existence of exponentially growing “potential” terms. Then, Belinskii and Khalatnikov have claimed that a scalar field ϕ exponentially coupled with the Maxwell field restores Mixmaster oscillations in homogeneous spacetimes [5]. Berger also suggested that if there exists a “potential”

$$V(\phi) = \mathcal{A}^2 e^{\alpha\phi} + \mathcal{B}^2 e^{-\beta\phi}, \quad \alpha, \beta > 0, \quad (3.1)$$

where \mathcal{A} and \mathcal{B} are functions, then Mixmaster oscillations can be retained in both homogeneous and inhomogeneous spacetimes [8].

At first glance, their suggestions are plausible since it seems that Mixmaster cannot be avoided whenever $\phi \rightarrow \pm\infty$ as $t \rightarrow 0$ in Eq. (3.1). However, the circumstances are not so simple since the nonlinear gravitational effect may not necessarily be even for both terms in Eq. (3.1). In general, functions \mathcal{A} and \mathcal{B} are constructed by metric functions. Therefore, there is a possibility that the potential V vanishes asymptotically by the gravitational effect in \mathcal{A}^2 and \mathcal{B}^2 even if $\phi \rightarrow \pm\infty$ as $t \rightarrow 0$. As a result, such spacetimes are not Mixmaster but AVTD. Thus, the AVTD singularity is expected even though there is a “potential” term like Eq. (3.1) in inhomogeneous spacetimes.

It should be noted that this form of potential arises from the low energy effective superstring theory. It contains the dilaton which is non-minimally coupled with the Maxwell and axion field. From the unified theoretical point of view, it is believed that the distinction between gravity and matter fields cannot be maintained near spacetime singularities and the superstring theory is the most promising candidate of such unified theories [33]. Also, it is pointed out that the EMD system with a positive cosmological constant has new mechanism forming singularities [25]. Hence, it is important to investigate the behavior around the singularity in the EMDA system.

IV. EINSTEIN-MAXWELL-DILATON-AXION SYSTEM IN GOWDY SPACETIMES ON $T^3 \times R$

In this section, we introduce a standard model with additional assumptions. First, we adopt Gowdy spacetimes. Gowdy proposed spatially compact and inhomogeneous spacetimes with the following metric [15];

$$ds^2 = e^{\lambda(t,\theta)/2} t^{-1/2} (-dt^2 + d\theta^2) + R(t, \theta) \left[e^{-Z(t,\theta)} (dy + X(t, \theta) dz)^2 + e^{Z(t,\theta)} dz^2 \right]. \quad (4.1)$$

Gowdy spacetimes have two twist free spacelike Killing vectors $\partial/\partial y$ and $\partial/\partial z$. The spatial topology of the spacetimes is classified into three types, T^3 , $S^2 \times S^1$ and S^3 . We will consider only the simplest case, T^3 , since the isometry group action has no degenerate orbits and therefore Einstein's equations have no corresponding singularities [28]. Gowdy spacetimes are called *polarized* if $X \equiv 0$, and *non-polarized* if this condition is not met.

Properties of the metric (4.1) depend on whether ∇R is timelike, spacelike or null. When the metric Eq. (4.1) describes a cosmological model, i.e. ∇R is globally timelike and the spatial topology is T^3 (periodic in θ), one can take the function $R(t, \theta) = t$ without loss of generality by Gowdy's corner theorem if the spacetime is vacuum [15]. This fact be seen from Einstein's equations,

$$G_{tt} - G_{\theta\theta} = \ddot{R} - R'' = 0, \quad (4.2)$$

where dot and prime denote t and θ derivatives, respectively. Also in the EDA system, Eq. (4.2) holds [4]. In the Einstein-Maxwell system, Eq. (4.2) is not satisfied generically. However in the case that the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ has only the following components [26];

$$F_{ty} = \dot{\omega}, \quad F_{\theta y} = \omega', \quad F_{tz} = \dot{\chi}, \quad F_{\theta z} = \chi',$$

Eq. (4.2) is satisfied. This situation does not change even if the dilaton field is exponentially coupled with the Maxwell field. Thus, in the EMDA system, we can put $R(t, \theta) = t$. Hereafter, we will choose this gauge. In this case, Gowdy spacetimes have spacelike singularities at $t = 0$ [15].

The action of the EMDA theory is

$$\begin{aligned} S &= \int d^4x \sqrt{-g} \left[- {}^{(4)}R + e^{-2a\phi} F^2 + 2(\nabla\phi)^2 + \frac{1}{3}e^{-4a\phi} H^2 \right] \\ &= -\frac{1}{2}t \left[\ddot{\lambda} - \lambda'' + \dot{Z}^2 - Z'^2 + e^{-2Z} \left(\dot{X}^2 - X'^2 \right) \right] \\ &\quad - 2e^{-2a\phi} \left[(e^{-Z} X^2 + e^Z) (\dot{\omega}^2 - \omega'^2) + e^{-Z} (\dot{\chi}^2 - \chi'^2) - 2Xe^{-Z} (\dot{\omega}\dot{\chi} - \omega'\chi') \right] \\ &\quad - 2t \left(\dot{\phi}^2 - \phi'^2 \right) - \frac{1}{2}e^{4a\phi} t (\dot{\kappa}^2 - \kappa'^2), \end{aligned} \quad (4.3)$$

where $\phi = \phi(t, \theta)$ is the dilaton field, $H = dB \equiv -\frac{1}{2}e^{4a\phi} * d\kappa(t, \theta)$ is the three-index antisymmetric tensor field dual to the axion field κ , and a is a coupling constant [33]. The functions are periodic in θ ($0 \leq \theta \leq 2\pi$) because the spatial topology is T^3 . [However, the spatial compactness is not relevant to our analysis.] In the case of the effective superstring theory, $a = 1$. If we neglect the axion field, $a = \sqrt{3}$ gives Kaluza-Klein theory. If $a = 0$, the dilaton and axion fields are decoupled with the gravity.

Note that the metric function λ is decoupled with other functions, and that λ appears only in the Hamiltonian and momentum constraints. This is the primary advantage of Gowdy spacetimes. Now, let us focus on evolution equations. We can calculate the metric function λ by evaluating the integral of λ' from 0 to 2π after obtaining other functions.

V. REVIEW OF THE FUCHSIAN ALGORITHM

The above system is very complicated and gives rise to highly nonlinear partial differential equations which are nontrivially coupled with the matter fields. We would like to construct AVTD singular solutions of such complicated equations which are parametrized by as many arbitrary functions as possible. In order to do this, we adopt the Fuchsian algorithm developed by Kichenassamy and Rendall [22]. One of the advantages of this algorithm is applicable to general (nonlinear and/or singular) partial differential equations (PDEs). Another advantage of the Fuchsian algorithm is applicable to arbitrary spatial dimension. [Spatial dimension in Gowdy spacetimes is one since these spacetimes have two spacelike Killing vector fields.] This fact suggests that the Fuchsian algorithm is applicable to spacetimes with fewer or no symmetry [31,1].

In this section, we briefly review the Fuchsian algorithm [22,19,21]. Let us consider a PDE system

$$F[u(t, x^\alpha)] = 0. \quad (5.1)$$

Generically, u can have any number of components. Here, we will assume that the PDE is singular with respect to the argument t . The Fuchsian algorithm consists of three steps as follows:

Step 1 Identify the leading (singular) terms $u_0(t, x)$ which is a part of the desired expansion for u . This means that the most singular terms cancel each other when $u_0(t, x)$ is substituted in Eq. (5.1).

Step 2 Introduce a renormalized unknown function $v(t, x)$, which is given by

$$u = u_0 + t^m v. \quad (5.2)$$

If $u_0 \sim t^k$, we should set $m = k + \varepsilon$, where $\varepsilon > 0$. Thus, v is a regular part of the desired expansion for u .

Step 3 Obtain a *Fuchsian system* for v by substituting Eq. (5.2) in Eq. (5.1). That is,

$$(D + A)v = t^\varepsilon f(t, x, v, \partial_x v), \quad (5.3)$$

where $D \equiv t\partial_t$ and A is a matrix which is independent of t and $\varepsilon > 0$. f can be assumed to be analytic in all of arguments except t and continuous in t . Note that Eq. (5.3) is a (*singular*) PDE system for the *regular* function v .

Roughly speaking, the Fuchsian algorithm is a transformation from finding singular solutions of the original equations (5.1) into finding regular solutions of the Fuchsian equations (5.3) which may be singular even if the original equations are regular. Once we have the Fuchsian system, we can show the existence of a unique solution for prescribed singular part u_0 by the following theorem.

Theorem 1 (Theorem 3 of Ref. [22]) *Let us consider a system Eq. (5.3), where A is an analytic matrix near $x = 0$, such that $\|\sigma^A\| \leq C$ for $0 < \sigma < 1$ (boundedness condition) and f is analytic in space x and continuous in time t . Then the system (5.3) has exactly one solution which is defined near $x = 0$ and $t = 0$, and which is analytic in space and continuous in time, and tend to zero as $t \rightarrow 0$.* \square

Applying the Fuchsian algorithm to our problem of Gowdy spacetimes in the EMDA system, the procedure is summarized as follows:

1. Solve VTD equations as ordinary differential equations with respect to t . Solutions obtained such a way are the leading (singular) terms of the desired formal solutions.
2. Substitute the formal solutions into full field equations. Here, make integral constants arbitrary functions of spatial arguments x .
3. Obtain a Fuchsian system for unknown functions and evaluate every eigenvalues of A .
4. Apply Theorem 1 to the system.

If every eigenvalues of A are non-negative, the boundedness condition in Theorem 1 holds. Then, renormalized unknown functions must vanish as $t \rightarrow 0$ by Theorem 1. Therefore, the only singular terms which are solutions to VTD equations remain and they are solutions to full field equations at $t = 0$. Thus, we have AVTD singular solutions. Essentially, the above argument is a singular version of the Cauchy-Kowalevskaya theorem [1,22].

VI. ASYMPTOTIC BEHAVIOR IN GOWDY SPACETIMES

We are now ready to examine the singularity in the EMDA system. The EMDA system with Gowdy symmetry, however, has six functions ($Z, X, \omega, \chi, \phi, \kappa$), and it is very complicated to treat all of them at the same time. Hence, we will reduce the system by selecting four functions from them for simplicity. As a selection rule, it is important to leave exponential coupling between the dilaton and other fields since, as mentioned in Sec.III, an essential part of our analysis is whether or not “potential” terms in Einstein’s equations exponentially grow as $t \rightarrow 0$ [7,8]. Therefore, we will consider three cases, which are the non-polarized EMD, the non-polarized EDA and the polarized EMDA.

A. non-polarized EMD ($\chi = 0, \kappa = 0$)

First, let us consider the case that the dilaton field is exponentially coupled with the Maxwell field. Varying the action (4.3) with respect to (Z, X, ω, χ) , we obtain evolution equations,

$$D^2 Z - t^2 Z'' + t^2 e^{-2Z} (\dot{X}^2 - X'^2) + 2te^{-2a\phi} (e^{-Z} X^2 - e^Z) (\dot{\omega}^2 - \omega'^2) = 0, \quad (6.1)$$

$$D^2 X - t^2 X'' - 2t^2 (\dot{X}\dot{Z} - X'Z') - 4te^{-2a\phi+Z} X (\dot{\omega}^2 - \omega'^2) = 0, \quad (6.2)$$

$$D^2 \omega - t^2 \omega'' - t^2 \left[2a\dot{\phi} + \frac{1}{t} - \frac{(2\dot{X} - \dot{Z}X)Xe^{-Z} + \dot{Z}e^Z}{e^{-Z}X^2 + e^Z} \right] \dot{\omega} \\ + t^2 \left[2a\phi' - \frac{(2X' - Z'X)Xe^{-Z} + Z'e^Z}{e^{-Z}X^2 + e^Z} \right] \omega' = 0, \quad (6.3)$$

$$D^2 \phi - t^2 \phi'' + ate^{-2a\phi} (e^{-Z} X^2 + e^Z) (\dot{\omega}^2 - \omega'^2) = 0. \quad (6.4)$$

Following the procedure explained in the previous section, we have VTD equations by dropping spatial derivative terms from Eqs.(6.1)-(6.4).

$$D^2 Z + t^2 e^{-2Z} \dot{X}^2 + 2te^{-2a\phi}(e^{-Z} X^2 - e^Z) \dot{\omega}^2 = 0, \quad (6.5)$$

$$D^2 X - 2t^2 \dot{X} \dot{Z} - 4te^{-2a\phi+Z} X \dot{\omega}^2 = 0, \quad (6.6)$$

$$D^2 \omega - t^2 \left\{ 2a\dot{\phi} + \frac{1}{t} - \frac{(2\dot{X} - \dot{Z}X)Xe^{-Z} + \dot{Z}e^Z}{e^{-Z}X^2 + e^Z} \right\} \dot{\omega} = 0, \quad (6.7)$$

$$D^2 \phi + ate^{-2a\phi}(e^{-Z} X^2 + e^Z) \dot{\omega}^2 = 0. \quad (6.8)$$

According to Definition 1, solutions to these equations are AVTD if they exist and are also solutions to full Einstein's equations as $t \rightarrow 0$.

Since we expect Kasner-like solutions, $Z \approx Z_0 \ln t$, $X \approx X_0$ and $\phi \approx \phi_0 \ln t$ are chosen as the leading terms for Z , X and ϕ , respectively. Substituting them into Eq. (6.7), we have a leading term of ω . Thus, we can construct a family of formal solutions to full Einstein's equations as,

$$Z = Z_0(\theta) \ln t + Z_1(\theta) + t^\epsilon \alpha(t, \theta), \quad (6.9)$$

$$X = X_0(\theta) + t^{2Z_0} (X_1(\theta) + \beta(t, \theta)), \quad (6.10)$$

$$\omega = \omega_0(\theta) + t^{Z_0+2a\phi_0+1} (\omega_1(\theta) + \gamma(t, \theta)), \quad (6.11)$$

$$\phi = \phi_0(\theta) \ln t + \phi_1(\theta) + t^\epsilon \delta(t, \theta), \quad (6.12)$$

where $Z_0 > 0$, $Z_0 + 2a\phi_0 + 1 > 0$, $\epsilon > 0$. The first and second inequalities are given by Eqs. (6.6) and (6.7). α , β , γ and δ are renormalized unknown functions. Note that when we put $\alpha = \beta = \gamma = \delta = 0$, Eqs. (6.9)-(6.12) are exact solutions to VTD equations as $t \rightarrow 0$. Substituting the formal solutions (6.9)-(6.12) into full Einstein's equations (6.1)-(6.4), we can obtain the following system,

$$(D + A)\vec{u} = \vec{f}(t, \theta, \vec{u}, \vec{u}'), \quad (6.13)$$

where

$$\vec{u} = (\alpha, D\alpha, t\alpha', \beta, D\beta, t\beta', \gamma, D\gamma, t\gamma', \delta, D\delta, t\delta')^T. \quad (6.14)$$

The matrix A has the 12×12 components,

$$A = \begin{pmatrix} A_Z & 0 & 0 & 0 \\ 0 & A_X & 0 & 0 \\ 0 & 0 & A_\omega & 0 \\ 0 & 0 & 0 & A_\phi \end{pmatrix},$$

where,

$$A_Z = \begin{pmatrix} 0 & -1 & 0 \\ \epsilon^2 & 2\epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_X = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 2Z_0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_\omega = \begin{pmatrix} 0 & -1 & 0 \\ 0 & Z_0 + 2a\phi_0 + 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\phi = \begin{pmatrix} 0 & -1 & 0 \\ \epsilon^2 & 2\epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can easily verify that eigenvalues of A are 0, $2Z_0$, $Z_0 + 2a\phi_0 + 1$ and ϵ , which are non-negative. Then, we conclude that the boundedness condition of Theorem 1 holds. Next, we examine the leading parts in components of the vector

\vec{f} .

$$\vec{f} = \begin{pmatrix} 0 \\ X_0'^2 e^{-2(Z_1+t^\epsilon\alpha)} t^{2-2Z_0-\epsilon} + 2X_0^2 \omega_0'^2 e^{-(2a(\phi_1+t^\epsilon\delta)+Z_1+t^\epsilon\alpha)} t^{1-Z_0-2a\phi_0-\epsilon} + \dots \\ t(\alpha + D\alpha)' \\ 0 \\ -2X_0'(Z_0' \ln t + Z_1' + t^\epsilon\alpha) t^{2-2Z_0} - 4X_0 \omega_0'^2 e^{-2a(\phi_1+t^\epsilon\delta)+Z_1+t^\epsilon\alpha} t^{1-Z_0-2a\phi_0} + \dots \\ t(\beta + D\beta)' \\ 0 \\ X_0^{-2} e^{Z_1+t^\epsilon\alpha} \{ \omega_0'' - [2a(\phi_0' \ln t + \phi_1' + t^\epsilon\delta') + 1] \omega_0' \} t^{1-Z_0-2a\phi_0} + \dots \\ t(\gamma + D\gamma)' \\ 0 \\ aX_0^2 \omega_0'^2 e^{-(2a(\phi_1+t^\epsilon\delta)+Z_1+t^\epsilon\alpha)} t^{1-Z_0-2a\phi_0-\epsilon} + \dots \\ t(\delta + D\delta)' \end{pmatrix}.$$

The dots represent the terms which are lower order with respect to t and are not important for our discussion since the power of these terms is necessarily positive if conditions $Z_0 > 0$ and $Z_0 + 2a\phi_0 + 1 > 0$ hold. Positivity of the power of t in components of the vector \vec{f} gives us sufficient conditions to obtain the AVTD solutions. Finally, we apply Theorem 1 to this system and then, we obtain the following theorem.

Theorem 2 *Let $Z_i(\theta)$, $X_i(\theta)$, $\omega_i(\theta)$, $\phi_i(\theta)$, where $i = 0, 1$, be real analytic functions and $0 < Z_0 < 1$ and $-1 < Z_0 + 2a\phi_0 < 1$ for $0 \leq \theta \leq 2\pi$. Then, for field equations (6.1)-(6.4) in the EMD system, there exists a unique solution with the form (6.9)-(6.12), where α , β , γ and δ tend to zero as $t \rightarrow 0$. \square*

Theorem 2 means that T^3 Gowdy spacetimes in the EMD system have AVTD singularities in “general” in the sense that the singular solutions have maximal number (i.e. eight) of arbitrary functions (Z_0 , X_0 , ω_0 , ϕ_0 , Z_1 , X_1 , ω_1 , ϕ_1).

In components of the vector \vec{f} , terms $t^{1-Z_0-2a\phi_0}$ and t^{2-2Z_0} give us upper bound for Z_0 and $Z_0 + 2a\phi_0$ in Theorem 2. However, since these terms are always multiplied by X_0' and ω_0' , the upper bound can be removed if $X_0' = 0$ and $\omega_0' = 0$. Hence, we obtain the following corollary:

Corollary 1 *Let $Z_i(\theta)$, $X_i(\theta)$, $\omega_i(\theta)$, $\phi_i(\theta)$, where $i = 0, 1$, be real analytic functions, and assume $0 < Z_0$, $-1 < Z_0 + 2a\phi_0$ and $X_0' = \omega_0' = 0$ for $0 \leq \theta \leq 2\pi$. Then, for field equations (6.1)-(6.4) in the EMD system, there exists a unique solution with the form (6.9)-(6.12), where α , β , γ and δ tend to zero as $t \rightarrow 0$. \square*

Note that this is a “non-generic” case in the sense that two of arbitrary eight functions are constant in θ .

B. non-polarized EDA ($\omega = 0$, $\chi = 0$)

Now, we will consider the case that the axion field is exponentially coupled with the dilaton field. In the similar way to Sec.VIA, we have evolution equations by varying the action (4.3) with respect to (Z, X, ϕ, κ) .

$$D^2 Z - t^2 Z'' + t^2 e^{-2Z} (\dot{X}^2 - X'^2) = 0, \quad (6.15)$$

$$D^2 X - t^2 X'' - 2t^2 (\dot{X}\dot{Z} - X'Z') = 0, \quad (6.16)$$

$$D^2 \phi - t^2 \phi'' - \frac{a}{2} t^2 e^{4a\phi} (\dot{\kappa}^2 - \kappa'^2) = 0, \quad (6.17)$$

$$D^2 \kappa - t^2 \kappa'' + 4at^2 (\dot{\phi}\dot{\kappa} - \phi'\kappa') = 0. \quad (6.18)$$

Note that a pair of PDEs for gravity (6.15) and (6.16) is decoupled with those for scalar fields (6.17) and (6.18). Furthermore, Eqs. (6.15) and (6.16) have equivalent form to Eqs. (6.17) and (6.18) as simultaneous PDEs. These facts are known as “mirror images” in string cosmologies and used to construct new nontrivial solutions [23].

Dropping the spatial derivative terms from Eqs. (6.15)-(6.18), VTD equations are obtained as follows:

$$D^2 Z + t^2 e^{-2Z} \dot{X}^2 = 0, \quad (6.19)$$

$$D^2 X - 2t^2 \dot{X}\dot{Z} = 0, \quad (6.20)$$

$$D^2 \phi - \frac{a}{2} t^2 e^{4a\phi} \dot{\kappa}^2 = 0, \quad (6.21)$$

$$D^2 \kappa + 4at^2 \dot{\phi}\dot{\kappa} = 0. \quad (6.22)$$

We can easily solve these equations as $t \rightarrow 0$ and find formal solutions to full Einstein's equations.

$$Z = Z_0(\theta) \ln t + Z_1(\theta) + t^\epsilon \alpha(t, \theta), \quad (6.23)$$

$$X = X_0(\theta) + t^{2Z_0}(X_1(\theta) + \beta(t, \theta)), \quad (6.24)$$

$$\phi = \phi_0(\theta) \ln t + \phi_1(\theta) + t^\epsilon \gamma(t, \theta), \quad (6.25)$$

$$\kappa = \kappa_0(\theta) + t^{-4a\phi_0}(\kappa_1(\theta) + \delta(t, \theta)), \quad (6.26)$$

where $Z_0 > 0$, $-2a\phi_0 > 0$ and $\epsilon > 0$. These restrictions for Z_0 and ϕ_0 given by Eqs. (6.20) and (6.22), respectively.

The system for regular functions α , β , γ and δ in the EDA system is given by Eqs. (6.13) and (6.14) by replacing the 12×12 matrix A by

$$A = \begin{pmatrix} A_Z & 0 & 0 & 0 \\ 0 & A_X & 0 & 0 \\ 0 & 0 & A_\phi & 0 \\ 0 & 0 & 0 & A_\kappa \end{pmatrix},$$

where

$$A_\kappa = \begin{pmatrix} 0 & -1 & 0 \\ 16a^2\phi_0^2 & -4a\phi_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Eigenvalues of A are 0, $2Z_0$, $-4a\phi_0$ and ϵ , and all of them are non-negative. These facts imply that the boundedness condition in Theorem 1 holds. After long calculation, the leading parts in components of the vector \vec{f} is obtained as follows.

$$\vec{f} = \begin{pmatrix} 0 \\ X_0'^2 e^{-2(Z_1+t^\epsilon\alpha)} t^{2-2Z_0-\epsilon} + \dots \\ t(\alpha + D\alpha)' \\ 0 \\ [X_0'' - 2X_0'(Z_0' \ln t + Z_1' + t^\epsilon\alpha')] t^{2-2Z_0} + \dots \\ t(\beta + D\beta)' \\ 0 \\ -\frac{a}{2}\kappa_0'^2 e^{4a(\phi_1+t^\epsilon\gamma)} t^{2+4a\phi_0-\epsilon} + \dots \\ t(\gamma + D\gamma)' \\ 0 \\ [\kappa_0'' + 4a\kappa_0'(\phi_0' \ln t + \phi_1' + t^\epsilon\gamma')] t^{2+4a\phi_0} + \dots \\ t(\delta + D\delta)' \end{pmatrix}.$$

Since the power of t of other terms in \vec{f} is positive under conditions $Z_0 > 0$ and $-2a\phi_0 > 0$, and then they are not needed here. The AVTD behavior requires that the power of the leading parts of t is positive. Thus, we have the following theorem which states that non-polarized Gowdy spacetimes in the EDA system are AVTD in general.

Theorem 3 *Let $Z_i(\theta)$, $X_i(\theta)$, $\phi_i(\theta)$, $\kappa_i(\theta)$, where $i = 0, 1$, be real analytic and $0 < Z_0 < 1$ and $0 < -2a\phi_0 < 1$ for $0 \leq \theta \leq 2\pi$. Then, for field equations (6.15)-(6.18) in the EDA system, there exists a unique solution of the form (6.23)-(6.26), where α , β , γ and δ tend to zero as $t \rightarrow 0$.* \square

As seen from the vector \vec{f} , the upper bound for Z_0 and $-\phi_0$ in Theorem 3 are derived from terms multiplied by X_0' and κ_0' . Thus, we obtain the following corollary in the “non-generic” case.

Corollary 2 *Let $Z_i(\theta)$, $X_i(\theta)$, $\phi_i(\theta)$, $\kappa_i(\theta)$, where $i = 0, 1$, be real analytic, and assume $0 < Z_0$, $0 < -2a\phi_0$ and $X_0' = \kappa_0' = 0$, for $0 \leq \theta \leq 2\pi$. Then, for field equations (6.15)-(6.18) in the EDA system, there exists a unique solution of the form (6.23)-(6.26), where α , β , γ and δ tend to zero as $t \rightarrow 0$.* \square

C. polarized EMDA ($X = 0$, $\chi = 0$)

Finally, we will study the case of polarized Gowdy spacetimes in the EMDA system. As Berger has suggested, the existence of “potential” like Eq. (3.1) might lead to Mixmaster oscillations. Such a potential appears in the

EMDA system [See Eq. 6.29]. On the other hand, it has been shown that vacuum polarized T^3 Gowdy spacetimes are AVTD [20]. Which behavior is realized in the EMDA system, AVTD or Mixmaster?

Varying the action (4.3) with respect to four functions $(Z, \omega, \phi, \kappa)$, we have evolution equations as follows:

$$D^2 Z - t^2 Z'' - 2te^{-2a\phi+Z}(\dot{\omega}^2 - \omega'^2) = 0, \quad (6.27)$$

$$D^2 \omega + t^2 \left(\dot{Z} - 2a\dot{\phi} - \frac{1}{t} \right) \dot{\omega} - t^2 \omega'' - t^2 (-2a\phi' + Z') \omega' = 0, \quad (6.28)$$

$$D^2 \phi - t^2 \phi'' + ate^{-2a\phi+Z}(\dot{\omega}^2 - \omega'^2) + \frac{1}{2}at^2 e^{4a\phi}(\dot{\kappa}^2 - \kappa'^2) = 0, \quad (6.29)$$

$$D^2 \kappa - t^2 \kappa'' + 4at^2(\dot{\kappa}\dot{\phi} - \kappa'\phi') = 0. \quad (6.30)$$

Dropping the spatial derivative terms from Eqs. (6.27)-(6.30), following VTD equations are obtained as follows:

$$D^2 Z - 2te^{-2a\phi+Z}\dot{\omega}^2 = 0, \quad (6.31)$$

$$D^2 \omega + t^2 \left(\dot{Z} - 2a\dot{\phi} - \frac{1}{t} \right) \dot{\omega} = 0, \quad (6.32)$$

$$D^2 \phi + ate^{-2a\phi+Z}\dot{\omega}^2 + \frac{1}{2}at^2 e^{4a\phi}\dot{\kappa}^2 = 0, \quad (6.33)$$

$$D^2 \kappa + 4at^2 \dot{\kappa}\dot{\phi} = 0. \quad (6.34)$$

Similarly to the case of the non-polarized EDA system, we can solve these equations as $t \rightarrow 0$. Thus, a family of formal solutions of this system can be obtained as follows:

$$Z = Z_0(\theta) \ln t + Z_1(\theta) + t^\epsilon \alpha(t, \theta), \quad (6.35)$$

$$\omega = \omega_0(\theta) + t^{2a\phi_0 - Z_0 + 1}(\omega_1(\theta) + \beta(t, \theta)), \quad (6.36)$$

$$\phi = \phi_0(\theta) \ln t + \phi_1(\theta) + t^\epsilon \gamma(t, \theta), \quad (6.37)$$

$$\kappa = \kappa_0(\theta) + t^{-4a\phi_0}(\kappa_1(\theta) + \delta(t, \theta)), \quad (6.38)$$

where $2a\phi_0 - Z_0 + 1 > 0$, $-4a\phi_0 > 0$ and $\epsilon > 0$. Eqs. (6.32) and (6.34) restrict the values of Z_0 and ϕ_0 . By substituting Eqs. (6.35)-(6.38) into Eqs. (6.27)-(6.30), the system in the polarized EMDA system is given by Eqs. (6.13) and (6.14) with A given by

$$A = \begin{pmatrix} A_Z & 0 & 0 & 0 \\ 0 & A_\omega & 0 & 0 \\ 0 & 0 & A_\phi & 0 \\ 0 & 0 & 0 & A_\kappa \end{pmatrix},$$

where

$$A_\omega = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 2a\phi_0 - Z_0 + 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\kappa = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -4a\phi_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can easily evaluate the eigenvalues of the matrix A as 0, $2a\phi_0 - Z_0 + 1$, $-4a\phi_0$ and ϵ . Since they are all non-negative, the boundedness condition in Theorem 1 hold. The leading parts in components of the vector \vec{f} are as follows:

$$\vec{f} = \begin{pmatrix} 0 \\ -2\omega_0'^2 e^{-2a(\phi_1+t^\epsilon\gamma)+Z_1+t^\epsilon\alpha} t^{1-2a\phi_0+Z_0-\epsilon} + \dots \\ t(\alpha + D\alpha)' \\ 0 \\ \{\omega_0'' + \omega_0'[-2a(\phi_0' \ln t + \phi_1' + t^\epsilon\gamma') + Z_0' \ln t + Z_1' + t^\epsilon\alpha']\} t^{1-2a\phi_0+Z_0} + \dots \\ t(\beta + D\beta)' \\ 0 \\ a\omega_0'^2 e^{-2a(\phi_1+t^\epsilon\gamma)+Z_1+t^\epsilon\alpha} t^{1-2a\phi_0+Z_0-\epsilon} + \frac{a}{2}\kappa_0'^2 e^{4a(\phi_1+t^\epsilon\gamma)} t^{2+4a\phi_0-\epsilon} + \dots \\ t(\gamma + D\gamma)' \\ 0 \\ [\kappa_0'' + 4a\kappa_0'(\phi_0' \ln t + \phi_1' + t^\epsilon\gamma')] t^{2+4a\phi_0} + \dots \\ t(\delta + D\delta)' \end{pmatrix}.$$

We do not need other terms since their power is positive under the conditions $2a\phi_0 - Z_0 + 1 > 0$ and $-4a\phi_0 > 0$. Thus, we have again the theorem which states that polarized Gowdy spacetimes in the EMDA system are AVTD in general even if there exists a “potential” as Eq. (3.1) in the system. This result is a counter-example of Berger’s suggestion [8].

Theorem 4 *Let $Z_i(\theta)$, $\omega_i(\theta)$, $\phi_i(\theta)$, $\kappa_i(\theta)$, where $i = 0, 1$, be real analytic, and $-1 < 2a\phi_0 - Z_0 < 1$ and $-1 < 2a\phi_0 < 0$ for $0 \leq \theta \leq 2\pi$. Then, for field equations (6.27)-(6.30) in the EMDA system, there exists a unique solution of the form (6.35)-(6.38), where α , β , γ and δ tend to zero as $t \rightarrow 0$.* \square

The leading terms in components of the vector \vec{f} always include factors ω'_0 and/or κ'_0 . Then, we can be weaken the conditions for Z_0 and ϕ_0 as similar to the cases of the EMD and the EDA systems if we put $\omega'_0 = \kappa'_0 = 0$.

Corollary 3 *Let $Z_i(\theta)$, $\omega_i(\theta)$, $\phi_i(\theta)$, $\kappa_i(\theta)$, where $i = 0, 1$, be real analytic, and assume $\omega'_0 = \kappa'_0 = 0$, $-1 < 2a\phi_0 - Z_0$ and $2a\phi_0 < 0$ for $0 \leq \theta \leq 2\pi$. Then, for field equations (6.27)-(6.30) in the EMDA system, there exists a unique solution of the form (6.35)-(6.38), where α , β , γ and δ tend to zero as $t \rightarrow 0$.* \square

VII. SUMMARY

Theorem 2, 3 and 4 state that T^3 Gowdy spacetimes in the EMDA system have AVTD singularities in “general” in the sense that these AVTD singular solutions depend on the maximal number of singular data. In the “non-generic” cases where some functions in the singular data are constant in θ , we have obtained corollary 1, 2 and 3. These results support BKL conjecture, that is, the dynamics of spatially different points effectively are decoupled from each other. However the spacetimes in our system do not show an oscillation of the Mixmaster spacetime near cosmological singularities for some parameter regions. In this sense, our system is a special case.

Let us compare our results with the results obtained by Grubišić-Moncrief and Kichenassamy-Rendall in the (non-) polarized and vacuum case [16,22]. The low-velocity condition derived by Grubišić-Moncrief and Kichenassamy-Rendall are equivalent to our condition $0 < Z_0 < 1$. If spacetimes are vacuum, then non-polarized T^3 Gowdy spacetimes are AVTD under this condition. Also, when $X \equiv 0$ (polarized), the spacetimes are AVTD without the low-velocity condition. In any case, our results clearly include Grubišić-Moncrief and Kichenassamy-Rendall’s when the matter fields vanish.

When we take the matter field into account, Theorem 4 and corollary 3 give counter-examples to Berger’s suggestion since the existence of a “potential” Eq.(3.1) does not necessarily cause the Mixmaster behavior [8]. We should, however, put another condition on the initial matter field in addition to the low-velocity condition to realize the AVTD behavior by the effect of the “potential”. Thus, Berger’s suggestion may be correct in the weak sense that the AVTD behavior is controlled by the “potential”.

We can verify that the Kretschmann invariant $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$ of all of our AVTD solutions blows up as $t^{-(Z_0^2+2\phi_0^2+3)}$. Thus, the AVTD singularity in the EMDA system is the curvature singularity. This indicates the validity of the strong cosmic censorship for T^3 Gowdy spacetimes in the EMDA system since the spacetimes cannot be extended beyond the singularity similar to the vacuum case [16,20,28,12].

To avoid complicated calculation we focused only on four functions among the six functions (Z , X , ω , χ , ϕ and κ) in the present paper. For a next step, we would like to investigate the full system where that all of six functions do not vanish. The power of the leading parts with respect to the argument t for the Fuchsian system is $2Z$, $-4a\phi$, $2 - 2Z$, $2 + 4a\phi$, $1 \pm 2a\phi \pm Z$ and/or $1 \pm 2a\phi \mp Z$ in any case, as seen from the action (4.3) of the EMDA theory. We can make all powers positive simultaneously. Thus, we expect that T^3 Gowdy spacetimes in the EMDA system would be AVTD at the singularity in general.

Another subject to consider is asymptotic behavior of $U(1)$ -symmetric spacetimes [29] which have only one spacelike Killing vector. Although Einstein’s equations for $U(1)$ -symmetric spacetimes are more complicated than those of T^3 Gowdy spacetime, the Fuchsian algorithm is applicable to such system since this algorithm is independent on dimension of space. Recently, the Fuchsian algorithm is used for vacuum polarized $U(1)$ -symmetric spacetimes and for non-symmetric spacetimes in the Einstein-scalar field system [19,1,2]. Our interest is to know whether non-polarized $U(1)$ -symmetric spacetimes in the EMDA system are AVTD or not.

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